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2939. Proposed by C. F. GUMMER, Queen's University.

Show that the determinant

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_n & -b_1 & 0 & \cdots & 0 \\ 0 & a_1 & \cdots & a_{n-1} & -b_2 & -b_1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & a_1 & -b_n & -b_{n-1} & \cdots & -b_1 \\ b_1 & b_2 & \cdots & b_n & a_1 & 0 & \cdots & 0 \\ 0 & b_1 & \cdots & b_{n-1} & a_2 & a_1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & b_1 & a_n & a_{n-1} & \cdots & a_1 \end{vmatrix}$$

has the property that every n-rowed determinant of the n left-hand columns is equal to its cofactor.

2940. Proposed by R. E. GILMAN, Brown University.

Find the average number of operations required to obtain m white balls from an urn containing p white and q black balls ($m < p$) as follows: For the first operation m balls are drawn simultaneously and examined, and such as are found to be black, are returned to the urn. Each subsequent operation is like the first save that the number of balls drawn is equal to the number replaced in the urn in the preceding operation.

NOTES

24. Problems discussed by Huygens.—In the last published volume (14, 1920) of *Oeuvres Complètes de Christian Huygens* (cf. *1921*, 79) many problems are discussed. Among them are the following four: (*a*) [pages 208–209] dated 1655, "Given the sum of the sides of a right angled triangle and the difference of the segments of the hypotenuse made by a perpendicular dropped from the right angle, to construct the triangle"; (*b*) [pages 271–272] dated 1657, "Given two circles and a straight line, to describe a circle tangent to the given circles and with an arc cut off by the given line so as to contain an angle equal to a given angle"; (*c*) [pages 498–500] dated 1662, "To inscribe a regular heptagon in a circle"; (*d*) [pages 521–523] dated 1666, "To find the integer which, when divided by three given integers, the remainders are three given integers."

Problem (*a*)—After algebraic analysis Huygens gives a resulting geometrical construction and proof. An earlier and different discussion, by synthetic geometry, was published in 1607 by M. Ghetaldi in his *Variorum Problematum Collectio*, Venice, 1607, pp. 25–27; see also Oughtred, *Clavis Mathematicae*, third edition, Oxford, 1652, pp. 86–87. The more general problem, "Given the vertical angle, the sum of the two including sides, and the difference of the segments of the base made by a perpendicular dropped from the vertex, to construct the triangle," was solved by Renaldini in his *De Resolutione et Compositione Mathematica*, Padua, 1668, pp. 319, 529, and by T. Simpson, in his *A Treatise of Algebra*, London, 1745, pp. 292–293.

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Problem (*b*)—This problem was proposed to Huygens by R. F. de Sluse, in a letter dated[1] October 23, 1657, and Huygens here shows merely how the solution of the problem may be reduced to the solution of a quadratic equation.

In a letter to Fermat, dated July 29, 1654, Pascal refers[2] to two problems which he had solved by drawing only circles and straight lines; the second of these problems was as follows: "De trois cercles, trois lignes, [trois] quelconques étant donnés, trouver un cercle qui, touchant les cercles et les points, laisse sur les lignes un arc capable d'angle donné." But the particular one of these problems which we are considering was proposed by Pascal to Sluse, who then passed it on to Huygens. Some indication of Sluse's discussion is given in *Oeuvres de Blaise Pascal* publiées par L. Brunschvieg, P. Boutroux, et F. Gazier (vol. 7, Paris, 1914, pages 246–247; see however pages[1] 233–252; also tome 3, 1908, p. 302).

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[1] *Oeuvres Complètes de Christian Huygens*, vol. 2, 1889, pp. 72 and 80.

[2] *Oeuvres de Fermat*, vol. 2, Paris, 1894, p. 298.

This problem is, however, only a particular case of the following which can be solved with ruler and compasses only: Describe a circle which shall cut three given coplanar circles under given angles. The first formulation and solution of this problem seems to have been by F. Neumann[2] in 1825. In 1826 Steiner referred to the problem as one for which he had a solution[3] but, so far as known, this was never published. Plücker's discussion and construction appeared[4] in 1827. The problem has also been dealt with in Adolf Anderssen's program[5] of 1864; and by Griffiths,[6] 1871–1874; by Fiedler,[7] 1882; by Laquière,[8] 1883; by Tarry,[9] 1889; by M. Fouché,[10] 1892; by Casey,[11] 1893; by Lachlan,[12] 1893; and by many others.

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[1] Pages 248–255 contain a letter "de Sluse a Brunetti." This same letter is given in *Oeuvres de Fermat*, vol. 2, 1894, pp. 315–320 as of "Fermat a Carcavi."

[2] F. Neumann, *Isis*, Berlin, 1826; see E. Kötter, "Die Entwickelung der synthetischen Geometrie," *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 5, 1901, p. 115.

[3] J. Steiner "Einige geometrische Betrachtungen," *Journal für die reine und angewandte Mathematik*, vol. 1, 1826, p. 163; *Jacob Steiner's Gesammelte Werke*, vol. 1, 1881, p. 21. Similarly for the following problem on these same pages: To describe a sphere which shall cut each of four given spheres S_1, S_2, S_3, S_4, respectively, under the angles α_1, α_2, α_3, α_4.

[4] *Annales de Mathématiques Pures et Appliquées*, vol. 18, pp. 43–45, August, 1827; also Plücker, *Analytisch-geometrische Entwickelungen*, Essen, vol. 1, 1828, pp. 120–122. See also *Nouvelles Annales de Mathématiques*, 1870, pp. 371–375.

[5] *Ueber die Aufgabe einen Kreis zu konstruiren, der drei gegebene Kreise unter den Winkeln α, β, γ schneidet*, Osterprogramm des königlichen Friedrich Gymnasiums, Berlin (?), 1864.

[6] J. Griffiths: (*a*) "On the problem of finding the circle which cuts three given circles at given angles," *Proc. London Math. Soc.*, vol. 3, pp. 269–278, 1871; he found also equations for the groups of circles cutting three given small circles on a sphere at given angles. (*b*) "On the cartesian equation of the circle which cuts three given circles at given angles," *Proc. London Math. Soc.*, vol. 5, pp. 33–35, 1874.

[7] W. Fiedler, *Cyklographie oder Construction der Aufgaben über Kreise und Kugeln*, Leipsic, 1882, pp. 169–172.

[8] E. M. Laquière, "Détermination et construction nouvelle du cercle qui coupe trois cercles sous trois angles donnés et de la sphère qui coupe quatre sphères sous des angles donnés," *Nouvelles Annales de Mathématiques*, 1883, vol. 42, pp. 348–352.

[9] G. Tarry, "Sur un problème classique," *Journal de Mathématiques Elémentaires* (Bourget), 1889, pp. 217–222; he considers also Steiner's problem of a sphere cutting four given spheres under given angles. The argument of that part of the article relating to circles is given in E. Rouché et C. de Comberousse, *Traité de Géométrie*, 6e édition, part 1, Paris, 1891, pp. 288–290; 7e éd., pp. 304–306.

[10] M. Fouché, "Sur les cercles qui touchent trois cercles donnés ou qui les coupent sous un angle donné," *Nouvelles Annales de Mathématiques*, 1892, pp. 346–349.

[11] J. Casey, *A Treatise on the Analytical Geometry of the Point, Line, Circle and Conic Sections*, second edition, Dublin, 1893, pp. 108–109.

[12] R. Lachlan, *An Elementary Treatise on Modern Pure Geometry*, London, 1893, pp. 239–241.

Problem (*c*)—In seeking to construct a regular inscribed heptagon Huygens was led to a consideration of the cubic equation

$$z^3 - 2z^2 - z + 1 = 0, \tag{1}$$

or, as reduced by the removal of its second term,

$$y^3 - \frac{7}{3}y - \frac{7}{27} = 0, \tag{2}$$

and the solution of this equation[1] he tried, apparently, to effect by the intersection of curves.

The editors of Huygens's *Oeuvres* note that when Schooten discussed this problem several years earlier,[2] he was led to the equation

$$x^6 - 7x^4 + 14x^2 - 7 = 0, \tag{3}$$

for the heptagon, and to

$$x^3 - x^2 - 2x + 1 = 0, \tag{4}$$

for the tetradecagon, where x represents in each case, for one root, the ratio of a side of the polygon to the radius of the circumscribed circle. Equation (4) appears also in connection with the heptagon problem itself, as we should study it, since the roots of this equation, $2 \cos (\pi/7)$, $-2 \cos (2\pi/7)$, $2 \cos (3\pi/7)$, are the abscissas of the vertices of a heptagon inscribed in a circle of radius 2.

These same equations were found earlier still, in an endeavor to solve our problem of the heptagon. Equation (4) occurs in work of an unknown Arab, who flourished[3] about 980 A.D., and was found also by Vieta (1540–1603), whose works, edited by Schooten,[4] were published in 1646. Equation (3) was given by Kepler in his *Harmonice Mundi Liber I*, Linz, 1619, where, in proposition[5] 45, considerable space is also devoted to a discussion of approximate constructions for the heptagon. Kepler was convinced that its construction with ruler and compasses alone was impossible.

In his *Treatise of Angular Sections*, Oxford, 1684, John Wallis found (p. 48) the equation $RccH = 7RccA - 14RqqAc + 7RqAqc - Aqqc$, that is, $R^6H = 7R^6A$

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[1] z is the length of the longer diagonal of the heptagon whose side is of unit length; $y = z - 2/3$.

[2] F. van Schooten: (*a*) *Exercitationum Mathematicarum*, Liber V, Leyden, 1657, pp. 467–473; (*b*) *Vyfde Bouck der Mathematische Oeffeningen*, Amsterdam, 1660, pp. 436–442.

[3] F. Woepcke, *L'Algèbre d'Omar Alkhayâmi*, Paris, 1851, pp. 126–127.

[4] *Francisci Vietæ Opera Mathematica*, Leyden, 1646, "protasis IV," pp. 359–364. The discussion here may be considered as indicating the following construction: "Given a circle with center A and diameter CAB; on CB produced take a point D such that $DB \cdot CD^2 = AD \cdot AB^2$; with D as center and AB as radius describe a circle to cut the given circle in E; then the arc BE is the seventh part of the circumference." Or, "to construct an isosceles triangle such that the length of the bisector of an exterior base angle (between the base and a side produced) is equal to the length of the side." (Compare *Nouvelles Annales de Mathématiques*, vol. 9, 1850, pp. 51–53, 151, 233–234.)

[5] Also *Joannis Kepleri Astronomi Opera Omnia*, edidit Ch. Frisch, vol. 5, 1864, pp. 101–107, 471–473. In this MONTHLY (*1914*, 14, 148) J. Q. McNatt and S. A. Joffe were led, in discussion of the regular inscribed heptagon, to equation (3), and they found, by Horner's method, $x^2 = .7530203962821\cdots$; so that x is approximately equal to $\sqrt{3}/2$, half the side of an inscribed equilateral triangle—a result to which we shall return later.

$- 14R^4A^3 + 7R^2A^5 - A^7$. Wallis then makes the following notable statement (p. 49): "The Seven Roots of this Equation; are, so many straight Lines from some one Point of the Circumference, to the Seven Angles of an inscribed Regular Heptagon." "Of these Roots (putting H. Affirmative,) the two least are Affirmative; the two next, are Negative; the two next to these, are again Affirmative; and, the greatest Negative." [In the equation above, R is the radius, and H is "the Subtense of the Septuple Arch," *i.e.*, $H = 2R \sin (\pi/7)$; one value of A is $2R \sin (\pi/7^2)$. In other words, Wallis's equation is really the equation which we obtain when we put $\pi/7^2$ for θ in the trigonometric identity, $- \sin 7\theta = 2^6 \sin^7 \theta - 7 \cdot 2^4 \sin^5 \theta + 14 \cdot 2^2 \sin^3 \theta - 7 \sin \theta$. Comparison may be made with Marqfoy's discussion referred to in the first footnote at the bottom of this page.]

Maria G. Agnesi found in her *Instituzioni Analitiche* (Milan, 1748, vol. 1, pp. 279–284; English edition by J. Colson, London, 1801, pp. 168–171) various equations to which she was led in the discussion of the problem of the regular inscribed heptagon. Two of these are: (*a*) $x^6 - 7r^2x^4 + 14r^4x^2 - 7r^6 = 0$ (where r is the radius of the circle and one value of x is the side of the heptagon) which is practically identical with equation (3) of Kepler and Schooten; (*b*) $8x^3 - 4rx^2 - 4r^2x + r^3 = 0$ (where one value of x is the distance of the side of the heptagon from the center of the circle, that is, $x = \cos (\pi/7)$); to be more accurate, the latter equation was given by Agnesi with incorrect signs before the second and fourth terms—which seems to have been first noticed by P. Frisi (*Opera*, Milan, vol. 1, 1782, p. 177). Grunert showed (*Archiv der Mathematik und Physik*, vol. 17, 1851, p. 360) that this equation reduced, by a transformation, to

$$u^3 - \frac{7}{12} r^2 u + \frac{7}{216} r^3 = 0,$$

an equation whose roots are minus one half of those of equation (2) multiplied by the radius r.

In this survey the next in chronological order would be Sir William Rowan Hamilton, who was led, 1862–1864, to equations we shall presently number (5) and (6); consideration of his work will be reserved for the concluding part of this topic.

In 1873 Affolter gave[1] a geometrical construction for the inscribed heptagon, assuming that a cardioid (or the general pedal curve of a circle) could be used, in one step, for the trisection of an angle.[2] He found the equation, identical with one of Hamilton's,

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[1] F. G. Affolter, "Construction des regulären Sieben- und Dreizehn-Ecks," *Mathematische Annalen*, vol. 6, pp. 593–595. Since $\cos 7\theta = 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta$ the equation $64x^7 - 112x^5 + 56x^3 - 7x - 1 = 0$ has for roots the cosines of the angles $2h\pi/7$; $h = 1, 2, \cdots, 7$. But $\cos (14\pi/7) = 1$, and $\cos (2h\pi/7) = \cos [2(7 - h)\pi/7]$; hence the equation of the seventh degree (compare Wallis) reduces to $8x^3 + 4x^2 - 4x - 1 = 0$, — (5) below. (This method of derivation is indicated by G. Marqfoy, *Nouvelles Annales de Mathématiques*, vol. 9, 1850, pp. 52–53).

[2] H. Hippauf, "Lösung des Problems der Trisection mittelst der Conchoïde auf circularen Basis," *Zeitschrift für mathematischen und naturwiss. Unterricht*, vol. 3, 1872, pp. 215–240; on page 537 of this volume, C. Albricht points out that he solved the same problem by the same

$$c^3 + \frac{1}{2}c^2 - \frac{1}{2}c - \frac{1}{8} = 0, \tag{5}$$

where $c = \cos(2h\pi/7)$, $h = 1, 2, 3$, which he reduced to Huygens's equation (2) by the substitution $2c = y - 1/3$.

Freeth showed[1] that regular polygons of 7, 21, 35, $\cdots$ sides could be constructed, by means of the curve $r = a[1 + 2 \sin(\theta/2)]$ which he called a nephroid, although in the previous year this term was applied by Proctor to the two-cusped epicycloid.

E. Pascal showed,[2] in 1887, that we could reduce the solution of the problems to construct polygons of 7 or 13 or 97 sides, to the solution of cubic equations which, for the heptagon, was found to be

$$x^3 + x^2 - 2x - 1 = 0, \tag{6}$$

whose roots are double of those of equation (5). Setting $x = y - 1/3$ he reduced this to equation (2), which he solved, as Descartes did,[3] by means of a circle and a parabola.

By trigonometric considerations Dickson was led[4] to equations (4) and (6) in papers of 1894 and 1914, when he proved the impossibility of the construction of the heptagon with ruler and compasses.

Feldblum employed[5] the Gaussian method to reduce the cyclotomic equation to equation (6), and proceeded to give a construction based upon the trisection of an angle. Compare Affolter's method above.

It is well known that any plane geometrical problem, which leads to a cubic equation, can be solved with ruler and compasses alone, if a parabola, or an ellipse, or a hyperbola is first drawn in the plane. Vahlen gave the construction for a regular heptagon by means of an ellipse.[6]

As an application of Lill's method[7] Adler[8] and Mitscherling[9] showed, by means of equation (6), that the problem of the construction of a regular heptagon could be carried through with the aid of two right-angled rulers.

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method in his "Die Fusspunktlinien der Kegelschnitte und ihre Anwendungen," Progr. Hermannstadt, 1863–1864.

[1] T. J. Freeth, "The nephroid, heptagon, etc.," *Proceedings of the London Mathematical Society*, vol. 10, pp. 130, 228–229, 1879.

[2] E. Pascal, "Costruzioni geometriche di tre poligoni regolari," *Giornale di matematiche*, Naples, vol. 25, pp. 82–97.

[3] R. Descartes, *Oeuvres de Descartes* publiées par C. Adam and P. Tannery, vol. 6, Paris, 1902, pp. 469f.

[4] L. E. Dickson in this MONTHLY: (*a*) "The inscription of regular polygons," *1894*, 299–300; (*b*) "On the construction of regular polygons of 7 and 9 sides," *1914*, 260–262.

[5] M. Feldblum, *Ueber Elementar-Geometrische Constructionen* (Diss. Göttingen), Warsaw, 1899, pp. 33–40.

[6] R. T. Vahlen, "Ueber kubische Konstructionen," *Archiv der Mathematik und Physik*, (3), vol. 3, pp. 116–117, 1902.

[7] E. Lill, "Résolution graphique des équations numériques de tous les degrés à une seule inconnue et description d'un instrument inventé dans ce but," *Nouvelles Annales de Mathématiques*, vol. 26, 1867, pp. 359–362.

[8] A. Adler, *Theorie der geometrischen Konstruktionen*, Leipsic, 1906, pp. 209–210; 262–263.

[9] A. Mitscherling, *Das Problem der Kreisteilung*, Leipsic, 1913, p. 73.

In the course of discussion of the inscription of a regular heptagon by means of an equilateral hyperbola, C. B. Haldeman was led[1] to $z^3 + Rz^2\sqrt{7} - R^3\sqrt{7} = 0$, where one root z is the side of the heptagon, and R is the radius of the circle. The equation whose roots are the squares of the roots of this equation is $z^3 - 7R^2z^2 + 14R^4z - 7R^6 = 0$, which is equivalent to one of the equations discussed by Agnesi. Elsewhere in this issue[2] Mr. Haldeman returns to the discussion of the heptagon and is led, as Wallis was in the seventeenth century, to the expression of $\sin 7A$ in terms of $\sin A$.

* * *

It is appropriate next to consider various approximations for the side of a regular heptagon. The first of these, so far as we know, occurs in a work on surveying by Heron of Alexandria who flourished between 50 B.C. and 100 A.D. Introductory to finding the area of the inscribed heptagon he gives a lemma[3] in which he states that the length of the perpendicular from the center of the circle on the side, r, of the regular inscribed hexagon is approximately equal to the side, s_7, of the regular inscribed heptagon; that is, $s_7 = (\sqrt{3}/2)r$, approximately. Of the basis for this statement of approximation, apparently well known, we are ignorant. But with Cantor we may surmise[4] a connection with a work "On a heptagon in a circle" which some Arabian writers attribute to Archimedes[5] (d. 212 B.C.). Heron does not formulate the approximate relation $s_7 = s_3/2$, where s_3 is the length of the side of an equilateral triangle inscribed in the circle, although this readily follows, since $s_3 = \sqrt{3}\,r$—as Heron shows.

In a book of geometrical constructions by Aboûl Wafâ (900–998) we find[6] a construction for the inscribed heptagon in which the side of the heptagon is definitely taken as half the side of the equilateral triangle inscribed in the same circle. It is notable that he adds[7]: "But this is an approximation and not an exact construction." Leonardo da Vinci (1452–1519) maintained the exactness of this construction[8] in his *Codice Atlantico*. The construction is called the Indian method (quaestio Indorum) by Jordanus Nemorarius (d. 1236) in his *De triangulis*[9] where the construction of the regular heptagon is considered on pages 42–44.

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[1] C. B. Haldeman, "Geometrical construction of the roots of a cubic, and inscription of a regular heptagon in a circle" in this MONTHLY, *1919*, 391.

[2] Department of Questions and Discussions.

[3] *Herons von Alexandria Vermessungslehre und Dioptra.* Griechisch und Deutsch von H. Schöne. Leipsic, 1903, pp. 54–55. See also *Heronis Alexandrini Opera*, vol. 4, *Geometrica*, ed. Heiberg, Leipsic, 1912, pp. 384–385.

[4] M. Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. 1, third ed., Leipsic, 1907, pp. 377, 307.

[5] J. L. Heiberg, *Quaestiones Archimedeae*, Copenhagen, 1879, p. 29.

[6] F. Woepcke, "Analyse et extrait d'un recueil de constructions géométriques par Aboûl Wafâ," *Journal Asiatique* (5), vol. 5, pp. 218–256, 309–359; reprinted, Paris, 1855, pp. 89.

[7] Many refer in recent times to A. G. Kästner, ("Unrichtige Verzeichnung des Siebenecks" in *Geometrische Abhandlungen*, Erste Sammlung, Göttingen, 1790, pp. 249–250) as the first to point out that the above construction for the heptagon was inexact.

[8] Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. 2, second edition, 1900, pp. 298–300 and 465.

[9] *Mitteilungen des Coppernicus-Vereins für Wissenschaft und Kunst zu Thorn*, herausgegeben von M. Curtze, vi. Heft, Thorn, 1887. See also Cantor, vol. 2, *l.c.*, p. 83.

This same method is the basis of constructions or statements in: (*a*) an anonymous work, *Geometria deutsch*, the time of composition of which is also unknown;[1] (*b*) the *Underweysung der messung mit dem zirckel uñ richtscheyt* . . ., 1525, by Albrecht Dürer[2] (1471–1528); (*c*) Vieta's *Pseudomesolabum et alia quaedam adiuncta capitula*,[3] 1596; (*d*) the *Geometrica Practica*, Rome, 1604, of Christoph Clavius (Schlüssel, 1537–1612), pages 407–409, with reference to one Carolus Marianus Cremonensis;[4] (*e*) D. Schwenter, *Geometriae Praticae Novae*, Nürnberg, 1618, p. 171; (*f*) the work of 1619, referred to above, by Kepler (1571–1630); and (*g*) a problem of C. Mydorge[5] (1585–1647).

With reference to Hamilton's exact equation (5) above, Günther explained[6] an interesting relation with the approximation we have been discussing. He substituted for the number in the last term of equation (6), namely for $1/8 = 64/512$, $65/512$, and found that the resulting equation had the root $\cos(2\pi/7) = (5/8)$ or $\sin(\pi/7) = (\sqrt{3}/4)$, which gives $s_7 = \sqrt{3}\, r/2$ as before. The angle subtended by a side of a regular heptagon at the center of its circumscribed circle is about 51° 25′ 43″; the angle subtended by the above approximation to the side is 51° 19′ 4″,—about 6 minutes too small.

The particular case when $n = 7$ of the following construction of Antoine de Ville,[7] revised by Bosse[8] for the side of a polygon of n sides, gives a result about

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[1] See Cantor, vol. 2, *l.c.*, p. 451.

[2] Compare H. Staigmüller, *Dürer als Mathematiker*, Stuttgart, 1891, pp. 5, 23. De Morgan wrote in *The Athenaeum*, London, September 12, 1863, in his "Notes on the history of perspective, no. VIII": "There is a very neat way of approximately dividing a circle into seven equal parts which I have traced through writers on perspective up to Albert Dürer, beyond whom I cannot carry it; I do not find it in books upon other kinds of practical geometry, though I am told it has re-appeared in a work of our own time. Half the line which joins the two intersections of the circles in Euclid's first proposition is very nearly the side of the inscribed heptagon: it is too small; but any one who would feel satisfied with 1*l.* as composition for a debt of 1*l.* 0s. 0½*d.* ought to be a trifle better satisfied with Albert Dürer's heptagon. An error of less than one inch in 40 feet is good drawing."

[3] Vieta, *Opera Mathematica*, 1646, pp. 283–284; compare Cantor, vol. 2, *l.c.*, p. 583.

[4] Compare Cantor, vol. 2, *l.c.*, p. 581; also Mitscherling, *l.c.*, p. 61.

[5] Compare Cantor, vol. 2, *l.c.*, p. 673.

[6] S. Günther, *Die Geometrischen Näherungskonstruktionen Abrecht Dürers*, Ansbach, 1886, p. 9.

[7] *Les Fortifications du chevalier Antoine de Ville, contenans la maniere de fortifier toute sorte de places* . . . Lyon, 1629 (achevé d'imprimer, 1er août, 1628); there were other editions in 1636, 1640 (some title pages have 1641), 1668 (at Paris), pp. 34–36, and 1672 (at Amsterdam).

[8] A. Bosse, *Traité des pratiqves geometrales et perspectives*, Paris, 1665, p. 62. De Ville joined C to D', the *first* point of division of the diameter AB, and thus determined E' on the circumference; the chord AE'' of an arc the double of AE' was, according to De Ville, approximately equal to the side of the polygon. The De Ville-Bosse construction is often ascribed to C. Renaldini, who gave it in his *De resolutione et compositione mathematica*, Padua, 1688, pp. 367–368. Renaldini considered this construction accurate in all cases, but his error was subsequently pointed out in Schultz, *Dissertatio de circuli divisione*, Königsberg, 1691. Another work, to which credit in connection with De Ville-Bosse's construction is often given, is N. Bion, *Traité de la construction et des instruments de mathématiques*, Paris, 1709; fourth edition, 1752, p. 22. The matter was also considered by Jean Bernoulli and Gabriel Cramer, *Jacobi Bernoulli, Basileensis, Opera*, vol. 2, Geneva, 1744, p. 765; compare A. G. Kästner, "Renaldins allgemeine unrichtige Regel, jedes ordentliche Vieleck im Kreise zu beschreiben," in *Geometrische Abhandlungen*, Erste Sammlung, Göttingen, 1790, pp. 266–281. Among scores of other references which might be given, I simply add: [Note by R. Wolf, on a ms. of 1716 by J. J. Sheuchzer, giving Bosse's construction] *Vierteliahrsschrift der Naturforschenden Gesellschaft in Zürich*, vol. 29, 1884, pp. 39–40; and E. Catalan, *Théorèmes et Problèmes de Géométrie Elémentaire*, 6e éd., Paris, 1879, pp. 278–279.

6 minutes too large, namely[1] 51° 31′ 5″. De Ville-Bosse's construction: On the diameter AB of a circle, construct an equilateral triangle ABC; divide AB into n equal parts; join D, the second point of division from A, to the point C, and produce CD through D to meet the circle in E; then AE is either exactly or approximately equal to the length of the side of the required regular inscribed polygon of n sides. De Ville remarks (page 29): "Ce problem se demonstre point: et au calcul il ne revient pas tout à fait precisément aux figures qui ont grand nombre de costez . . . il est à estimer pour la facilité et justesse plus grande que de tous les autres qui ont esté escrits pour ce sujet."

A much less inaccurate construction for the side of a regular polygon, discovered by Karl Bernard, Duke of Saxe-Weimar, was published in 1828 (*Correspondance Mathématique et Physique* (Quetelet), volume 4, p. 350). It is as follows: Let BOA at right angles to DOC be diameters of a circle in which a polygon of n sides is to be inscribed; divide BA into n equal parts and produce OA through A to M, and OC through C to N such that $AM = CN = BA/n$. Let P be the point nearer to A in which MN cuts the circle. Join this to the end, Q, of the third point of division from A along AB; then PQ is approximately the length of the side of the polygon for $n \geqq 5$. This construction leads (I. Ghersi, *Matematica Dilettevolle e Curiosa,* Milan, 1913, p. 414), for the circle of unit radius, to the formula

$$PQ = \frac{1}{n} \sqrt{[n^2 - 8n + 48 - (n - 6)\sqrt{(n^2 - 4n - 4)}]}.$$

A table of values $n = 5$ to $n = 30$ is given by Ghersi on page 415. For $n = 5$ the angle subtended at the center of the circle by PQ is too small by about 39′ 46″; for $n = 6$ the construction is exact; for $n = 7$, the angle is about 56″ too small; for $n = 8$, 2′ 47″ too small; the error increases to 3′ 48″ for $n = 10$ then decreases to 7″ for $n = 21$; for $n = 22$ to $n = 30$, PQ is too large and the subtended angle-excess varies from about 4″ to about 56″.

In the case of the De Ville-Bosse construction a formula corresponding to that given above for PQ was found by Gabriel Cramer (*l.c.*, 1744) to be $\sqrt{[n^2 + 4n + 16 - \sqrt{(n^4 + 8n^3 - 144n^2 + 512n - 512)}]}/\sqrt{(2n^2 - 4n + 8)}$. For $\sqrt{(n^4 + 8n^3 - 144n^2 + 512n - 512)}$ A. G. Kästner wrote (*Geometrische Abhandlungen,* Erste Sammlung, Göttingen, 1790, p. 271) $(n - 4)\sqrt{(n^2 + 16n - 32)}$.

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[1] Housel considered the accuracy of De Ville-Bosse's construction, for $n = 3$ to $n = 17$, in "Division pratique de la circonférence en parties égales," *Nouvelles Annales de Mathématiques,* vol. 12, 1853, pp. 77–79; for $n = 5$ to $n = 17$ the errors range from − 5′ 48″ to + 36′ 37″. Tempier proposed another construction which gave very much better results when n was large (*Nouvelles Annales* . . ., vol. 12, pp. 345–347, and vol. 13, p. 295). For $n > 8$ Tempier proposed the following construction: On the diameter AB of a circle, center O, construct an equilateral triangle ABC; divide the radius OB into n equal parts, the end of the fourth part from O, being D; join CO and CD, producing them to meet the circle in E and F respectively; then $\angle EOF = \delta$ is approximately the angle subtended by a side of the inscribed polygon. Tempier gave the formula

$$\sin \delta = \frac{12n + \sqrt{48n^2 - 512}}{3n^2 + 16}.$$

From this it was found that the formula was exact for $n = 12$, and that for $n = 8$ to $n =$ the error in δ was never greater than 2′ 48″, while for $n = 100$ it was only 53″.

Röber's construction of 1854 is discussed below.

In 1864 Henry Norton suggested[1]

$$\sin\frac{\pi}{7}=\frac{105}{242}=\frac{3\times 5\times 7}{11\times 11\times 2},$$

which for construction might be thought of as

$$\left(\frac{3\times 5}{2}\right)\Big/\left(\frac{11\times 11}{7}\right).$$

On this supposition $\sin(\pi/7) = .4338842975$ which is correct to six places of decimals.

In the same year S. M. Drach gave the following construction also correct to six places of decimals[2]: Let CD be the perpendicular from the vertex C of an equilateral triangle on the base AB; cut off $CM = (1/7 - 1/9)$ $(= 2/63)CD$ and make $\angle DMK = \frac{1}{8}\angle DCA$. Then CK (K is a point in AB) is the approximate length of the side of heptagon inscribed in the circle of radius AB.

E. Collignon applied the discussion of certain series of numbers to the construction of the regular heptagon (plate), in "Examen de certaines séries numériques et application a la géomëtrie," *Compte Rendu de l'Association Francaise pour l'Avancement des Sciences*, 1888, p. 22.

In 1891 A. A. Robb showed[3] that, "by the aid of Peaucellier-cells a machine may be constructed which will solve the problem of the inscription of a regular heptagon in a circle, within the limits of Euclidean geometry."

In 1892 Efremoff published a paper[4] giving $s_7 = (2\sqrt{5} - 1)/4$, which is readily constructed, and hence within a minute the central angle subtended by a side of the heptagon.

In 1894 J. D. Everett described a linkage (*Report of the . . . British Association for the Advancement of Science*, for 1894, pp. 559–561) for the automatic determination of the vertices of regular polygons.

The following construction of a fourteen-year-old youth leads to a chord subtending at the center of a circle an angle $2' \, 31.7''$ in excess of the true value (*Educational Times*, March, 1907, vol. 60, p. 143; also *Mathematical Questions and Solutions from the Educational Times*, new series, vol. 12, 1907, pp. 51–52). Given a circle with center O and radius $OA = r$. Let the tangent at A meet in C the circle with center A and radius equal to r. Let this latter circle be met in D by a circle with center C and radius CA. Let DC cut the given circle with center O in B. Then AB is the approximation to the side of the heptagon inscribed in the given circle.

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[1] "On fractional values for the heptagon and circle," *Philosophical Magazine*, (4), vol. 27, 1864, p. 281.

[2] "On Albert Dürer's heptagon-chord.—Second notice," *Philosophical Magazine*, (4), vol. 27, 1864, p. 320.

[3] *Mathematical Questions and Solutions from the "Educational Times,"* vol. 55, London, 1891, p. 61.

[4] D. Efremoff, ["Construction of the sides of the regular heptagon and nonagon to within 0.001 of the radius of the circumscribed circle," *Spaczinski's Messenger*], Odessa, no. 146, pp. 32–33 (in Russian).

The following pair of constructions of K. Hagge leads to remarkably close approximations ("Benaderingsconstructies voor den regelmatigen zevenhoek," *Wiskundig Tijdschrift*, vol. 10, 1913–1914, pp. 165–166). (*a*) With A as center construct circles of radius r, $2r$, $4r$, and $9r$. Let a straight line through A meet the circle of radius $2r$ in B and C, and the circle of $4r$ in D and E. Make $BF = CF = 7r$. Then DF is approximately the side of a regular heptagon inscribed in the circle of radius $9r$. The angle subtended by the side at A is only $9''$ in excess of what it should be. In the second construction the excess is only $0.44''$. (*b*) With center A construct circles of radii r, $3r$, $6r$, $7r$. Let a straight line through A meet the circle of radius $3r$ in B and B', of radius $6r$ in D and C, and of radius $7r$ in F and E. Make $CG = DG = 8r$, $FH = BG$ (H on circle of radius $7r$), $EK = EH$ (K on EF). With center A and radius $AK - r$, describe a circle. Then in this circle the chord of length $4r$ is approximately the side of a regular heptagon.

Ghersi gave other constructions (*l.c.*, 1913, pp. 416–417).

We conclude with Sir William Rowan Hamilton's discussion[1] of F. G. Röber's construction[2] of the heptagon. Sir William took up this question at the request of C. B. Wale, son-in-law of the Archbishop of Dublin, who asked his opinion of the value of a couple of essays of Röber. The following are extracts from Hamilton's letter to Wale (*l.c.*):

"A wish to gratify the archbishop and yourself was the first motive for my attempting to examine to some extent the Essays of Röber which you had the goodness to leave for me a few days ago, and to form some opinion on their value, unimportant as that opinion might be. But the Memoir on the ancient temples of Egypt (Röber, Dresden, 1854) has interested me profoundly. Indeed I have scarcely been able, since I opened it, to attend to anything else; and it led me into some long calculations which I have only just completed to my satisfaction. As I have paid no special attention to Egyptian Antiquities, nor meditated much on such mystical guesses as some have made at their inner meaning, the only point which I could hope to study usefully was the *geometrical discovery* announced in the first memoir, namely, 'the construction of the regular heptagon,' which the elder Röber appears to have *divined*, from the study of the ancient Temple Architecture.[3]

"I entered on the subject, perhaps with prejudice; for like most (if not all) modern geometers, I have been accustomed to hold, and indeed still do hold, that it is *impossible* to construct such a heptagon with the 'right line' and 'circle' *alone*. Yet, to my great surprise, I found no error in Röber's *numbers*; and on repeating the calculations on another plan, with Taylor's seven-figure logarithms, I found myself quite unable to pronounce whether Röber's *arc* erred in excess or in defect from the *exact* seventh part of the circumference; for that it *must err* I felt assured.

"It seemed, therefore, worth while to go much more closely to work; and laying tables entirely aside, to perform the *whole* of the work for myself, by *arithmetic alone*, and especially by extractions of *square roots*. And to be quite sure of a high degree of accuracy in the final result, I made it a rule to work with not fewer than *fifteen decimal places*, besides employing all verifications that I could think of in the progress of calculation, which, thus laboriously conducted, has covered many sheets of paper, and cost many hours on two or three successive days.

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[1] R. P. Graves, *Life of Sir William Rowan Hamilton*, vol. 3, Dublin, 1889, pp. 141–147 (including a letter, dated September 15, 1862, to C. B. Wale); 584–587 (including extracts of letters exchanged by Hamilton and De Morgan in 1862). "On Röber's construction of the heptagon" by W. R. Hamilton, *Philosophical Magazine*, (4), 27, 1864, pp. 124–132.

[2] First given, apparently, in his son's *Beiträge zur Erforschung der geometrischen Grundformen in den alten Tempeln Aegyptens, und deren Beziehung zur alten Naturkenntniss*, Dresden, 1854, pp. 15–16, and repeated in a posthumous work (p. 20) edited by the younger Röber, and published at Leipsic in 1861, entitled *Elementar-Beiträge zur Bestimmung des Naturgesetzes der Gestaltung und des Widerstandes, und Anwendung dieser Beiträge auf Natur und alte Kuntsgestaltung.*

[3] The subject of the Second Essay was "The Pyramids and Parthenon."

"At last, however, it is finished; and I should have no hesitation to commit myself publicly to the result, which is, technically expressed, that the natural *cosine* of the angle assigned by Röber's construction is, to thirteen decimals, 0.62349 00759 241, whereas the true cosine of the seventh part of four right angles deduced to a corresponding accuracy from a known cubic equation I find to be *a little less*, namely, 0.62348 98018 587. Admitting, though I do not believe it, that the two or three last of these of these decimals *may* be wrong after all the precautions taken, I am quite satisfied that the cosine of the *Egyptian Angle*—for really Röber seems to make it likely that the Egyptians did employ it—is *somewhat greater*, and that the cosine of the *true* or *geometrical* angle (for of course we can in geometrical *conception* divide the circumference into *any number* of equal parts) is somewhat less than 0.62349, and consequently that the supposed *Egyptian rule* of the *heptagon* is *not mathematically perfect*, though Röber seems to suppose it to be so. And if you should ever think me worth citing on the subject, I request you to bear in mind that such is *one* of the results of my investigation.

"But now let us turn the tables and inquire *how near* does the supposed ancient rule come to the truth? *How small*, in *practice*, is the *error* which *theory* pronounces to *exist*? And I answer that in *practice* the *error* does not exist *at all*. I do not think that *experiments* of measurement, &c., *could* be so conducted by men, at least in the present age, as to prove to sight that there was any error. For *practical purposes*, then, the elder of the Röbers, or the old Egyptian sage whose secrets he supposed himself to have divined, has *done the impossible*. . . . Yet the *practical success* of the rule is to me absolutely *wonderful*: and it is long since any discovery in science produced in me such a sensation of *surprise*. . . . I may thus recapitulate:

"(1) The (alleged) *Egyptian Rule* for the construction of the Regular Heptagon is, in rigour of theory, erroneous.

"(2) The *same rule* of construction of the Heptagon is, however, for *all practical purposes*, *perfect*.

"No artist of the present day, I feel sure, would undertake to divide a circle into 7 (seven) equal parts, with a superior, or even an *equal* accuracy, to that which the construction, if fully carried out, would give."

Hamilton's article on the subject describes Röber's diagram as "not very complex, and may even be considered elegant"; and then continues: "but the essential parts of the construction are sufficiently expressed by the following formulæ: in which p denotes a side of a regular pentagon; r, r' the radii of its inscribed and circumscribed circles; r'' the radius of a third circle, concentric with but exterior to both; p' a segment of the side p; and q, s, t, u, v five other derived lines. The result is, that in the right-angled triangle of which the inner diameter $2r$ is the hypotenuse, and u, v supplementary chords, the former chord (u) is *very nearly* equal to a side of a regular heptagon, inscribed in the interior circle; while the latter chord (v) makes with the diameter ($2r$) an angle ϕ, which is *very nearly* equal to the vertical angle of an isosceles triangle, whereof each angle at the base is triple of the angle at the vertex. In symbols, if we write $u = 2r \sin \phi$, $v = 2r \cos \phi$, then ϕ is found to be very nearly $= \pi/7$. It will be seen that the equations can all be easily constructed by right lines and circles alone, having in fact been formed as the expression of such a construction; and that the numerical ratios of the lines, including the numerical values of the sine and cosine of ϕ, can all be arithmetically computed, with a few extractions of square roots." The formulæ are: $(r + r')^2 = 5r^2$, $p^2 = 4(r'^2 - r^2)$, $(p'/p) = (r + \frac{1}{2}r')/(r + r')$, $q^2 = p^2 - p'^2$, $s^2 + ps = (p - q + r)^2$, $r''^2 = r^2 + s^2$, $t^2 = (r'r''/r)^2 - (r'' - r)^2$, $u^2 = 2r(2r - t)$, $v^2 = 2rt$, $u = 2r \sin \phi$, $v = 2r \cos \phi$.

Hamilton showed that

$$\sin \frac{\pi}{7} = 0.43388\ 37391\ 17558\ 1205,$$

that

$$\sin \phi = 0.43388\ 35812\ 03469\ 1138,$$

and that

$$\frac{\pi}{7} - \phi = +\ 0''.03615\ 23230\ 806.$$

> "Another way of rendering conceivable the extreme smallness of the practical error of that process, is to imagine a series of seven successive chords inscribed in a circle, according to the construction in question, and to inquire *how near* to the initial point the final point would be. The answer is, that the *last* point would fall *behind* the *first*, but only by about *half a second* (more exactly by $0''.506$). If then we suppose, for illustration, that these chords are *seven successive tunnels*, drawn *eastward* from station to station of the *equator of the earth*, the last tunnel would emerge to the *west* of the first station, but only by about *fifty feet*."

On September 15, 1862, Hamilton wrote to De Morgan as follows:

> "Are you aware that it is possible with only *six* extractions of *square roots* (*two* being those required for chord and tangent of 36°) to approach so nearly to the cosine of the *seventh part* of four right angles, or to the positive root of the *cubic*,
>
> $$2x^3 + x^2 - x = \tfrac{1}{4},$$
>
> that the *sevenfold arc* resulting shall err (in defect) by only about *half a second* from the true and whole circumference? so that for *all practical purposes* a *regular heptagon* can be constructed by the right line and circle *alone*."

De Morgan replied in part:

> "I can believe anything of *six* square roots. But I cannot believe that the Egyptians employed them—though they may have hit on a method which requires six square roots to represent it arithmetically."

Hamilton and De Morgan arrived independently at the result

$$\cos\frac{2\pi}{7} = 0.62348\ 98018\ 58733\ 53052\ 50$$

which on September 29 brought forth De Morgan's comment: "I take it that you and I are the only persons who know $\cos 2\pi/7$ to 22 places."

*

Problem (*d*)—Let a, b, and c be the given integers and n, o and p the corresponding remainders after their division into the required integer. Huygens finds that the integer $bcfn + acgo + abhp$ satisfies all the conditions, if f is an integer such that bcf divided by a gives the remainder unity, and similarly for g and acg/b, and for h and abh/c. As a numerical example Huygens refers to the Julian period, proposed by the noted J. J. Scaliger (1540–1609), and a rule given by P. de Billy; it was translated as, "A problem for finding the year of the Julian period by a new and very easy method," and published in *Philosophical Transactions of the Royal Society*, 1666, p. 324. The rule is as follows: Multiply the year of the solar cycle [n] by 4845 [$= bcf$], that of the moon [o] by 4200 [$= acg$], that of the cycle of indiction [p] by 6916 [$= abh$]. Then divide the sum of the products by 7980 [$= abc$] which is the Julian period. What remains from the division, neglecting the quotient, will be the year sought. Huygens takes $a = 28$ for the whole solar cycle; $b = 19$ for the whole lunar cycle; and $c = 15$ for the cycle of indiction. He takes also $n = 13$ for the number in a solar cycle of the particular year chosen; and similarly $o = 4$ for the lunar cycle and $p = 9$ for the cycle of indiction. Then

$$\begin{aligned}[bcfn + acgo + abhp]/abc &= [62985 + 16800 + 62244]/7980\\ &= 142029 \div 7980 = 17 + 6369/7980.\end{aligned}$$

Hence 6369 is the year of the Scaligerian period corresponding to the data.

This same problem is discussed in the section on Julian period in the article

"Calendar" of the *Encyclopædia Britannica,* ninth and eleventh editions. It is there shown that the year 1 of our era "had 10 for its number in the solar cycle, 2 in the lunar cycle, and 4 in the cycle of indiction"; hence "the question is therefore to find a number such, that when it is divided by the three numbers 28, 19, and 15 respectively the three quotients shall be 10, 2, and 4." Without recourse to the general discussion of Huygens it is found that the number is 4714.

Compare, F. K. Ginzel, *Handbuch der mathematischen und technischen Chronologie der Zeitrechnungsweise der Völker,* Leipsic, vol. 3, 1914, p. 182.

R. C. ARCHIBALD

SOLUTIONS.

499 (Geometry) [1916, 341; 1919, 414; 1920, 187]. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Find the surfaces all the plane sections of which are circles.

SOLUTION BY J. L. WALSH, Harvard University.

This problem was solved by Professor Cairns [*1920,* p. 187], who interpreted the term *surface* as *algebraic surface.* Inasmuch as there exist other interesting surfaces (analytic, regular, etc.), perhaps it is worth while to give a geometric proof of the theorem:

If every plane section of a point set S is a circle, then S is a sphere.

We restrict ourselves to the real domain, and interpret our hypothesis to mean that whenever a plane actually intersects S, the intersection is a circle, which may be a null circle.

If a section of S by a sphere Σ contains three distinct points, then the section also contains the entire circle through these points. For through these three points we can pass a plane which will cut S in a circle and also cut Σ in a circle. These two circles have three points in common and hence are identical.

Let P be any point of S. Transform S into a point set S' by means of an inversion in space with P as center of inversion. The point P is transformed into P', the point at infinity; we consider as is habitual in the geometry of inversion a single point P' to lie at infinity. Every plane section of S' corresponds to a plane section or a spherical section of S. Every straight line of points belonging to S' corresponds to a circle of points belonging to S; there is no straight line all of whose points belong to S, for a straight line lies in a plane and every plane section of S is a circle.

If S' consists merely of the point P' every plane which cuts S cuts it in the null-sphere P, and the theorem is proved. If S contains another point Q besides P every plane through these points will intersect S in a circle which is not a single point, and there will be more than one of these circles. If then besides P' another point Q' belongs to S', there must be lines through Q' belonging to S', more than one of them.

Likewise, if S contains two distinct points besides P these points must lie on a circle through P, the intersection of S with the plane which contains the three points. Therefore if S' contains two distinct points besides P' it must contain the line determined by these two points.

Now we have shown that if S' contains one point Q' distinct from P' it contains more than one line through this point. It contains then the line determined by any two points of two lines through Q', and therefore all the points of at least one plane through Q'.

But a plane belonging entirely to S' corresponds in the inversion to a sphere belonging to S. Thus we have proved that if S contains more than one point it contains all the points of a sphere. S cannot contain a sphere and any point outside of the sphere, for a plane through such a point intersecting the sphere would intersect S in more than a single circle.

The method of proof used in the present note easily yields the following theorem:

If a point set consists of more than two points and is such that every spherical section which contains three points of the set also contains the circle through those three points, then the set is a circle, a plane, a sphere, or every point of space.